Letter

Expected and unexpected routes to synchronization in a system of swarmalators

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Systems of oscillators whose internal phases and spatial dynamics are coupled, swarmalators, present diverse collective behaviors which in some cases lead to explosive synchronization in a finite population as a function of the coupling parameter between internal phases. Near the synchronization transition, the phase energy of the particles is represented by the *XY* model, and they undergo a transition which can be of the first order or the second depending on the distribution of natural frequencies of their internal dynamics. The first-order transition is obtained after an intermediate state (static wings phase wave state) from which the nodes, in cascade over time, achieve complete phase synchronization at a precise value of the coupling constant. For a particular case of natural frequencies distribution, a new phenomenon, the rotational splintered phase wave state, is observed and leads progressively to synchronization through clusters switching alternatively from one to two and for which the frequency decreases as the phase coupling increases.

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Studies on dynamical systems' synchronization (DSS) began in 1665 [1]. DSS describes the coherent behavior of interconnected dynamical systems and it is observed in areas like pacemaker's rhythmic evolution [2,3], flocking birds [4–6], and fish [7–9]. In physics, it appears, among others, in synchronous pendulum motion and oscillators with intermittent coupling [1,10–12].

Subsequent studies deepened DSS understanding, including Winfree's work on circadian rhythms [13] and Kuramoto's phase synchronization model [14]. Research expanded into complex networks, exploring multilayer structures with and without amplification effects [15,16].

Synchronization applications span various contexts [17] with coupled chaotic oscillators advancing the field [18–20]. Transition to synchronization in complex networks, particularly in systems with moving elements, attracted considerable interest and was the subject of extensive research [20–22].

Recently, a swarmalator model coupling the systems' spatial positions with the Kuramoto model's phase dynamics was introduced for mobile systems [23]. Swarmalators exhibit five main behavior patterns depending on the coupling strength between spatial and phase dynamics [23–25]. Lizarraga and Aguiar [26] improved O'Keeffe *et al.*'s model [23] by incorporating an external force, showing that the swarmalator system can synchronize or aggregate as the force's amplitude increases.

The explosive collective transition to synchronization in complex networks found by Gómez-Gardenes *et al.* can occur in scale-free Kuramoto networks [27]. Skardal and Arenas [28] revealed that disorder in the natural frequencies, depending on its amplitude, can also trigger explosive transitions. Similarly, other studies have uncovered explosive synchronization in adaptive and multilayer networks [29–31].

According to recent findings, two-dimensional (2D) swarmalator systems can exhibit first-order transition under attractive and repulsive interaction's effect in a swarmalator's multiplex [32]. Based on the swarmalators' 2D model proposed in Ref. [23], Sar *et al.* developed a 1D model showing that random pinning on swarmalators can lead a system to chaotic behavior [33]. Others have investigated the existence of antiphase synchronization between two swarmalator groups [32,34]. While these studies and others [35–37] explore the swarmalator model primarily through pairwise interactions, real-world systems often involve more complex dynamics. To bridge this gap, a new model incorporating higher-order interactions considers relationships beyond just two elements within a population [38]. This approach aims to capture better the complexity of interactions observed in real systems.

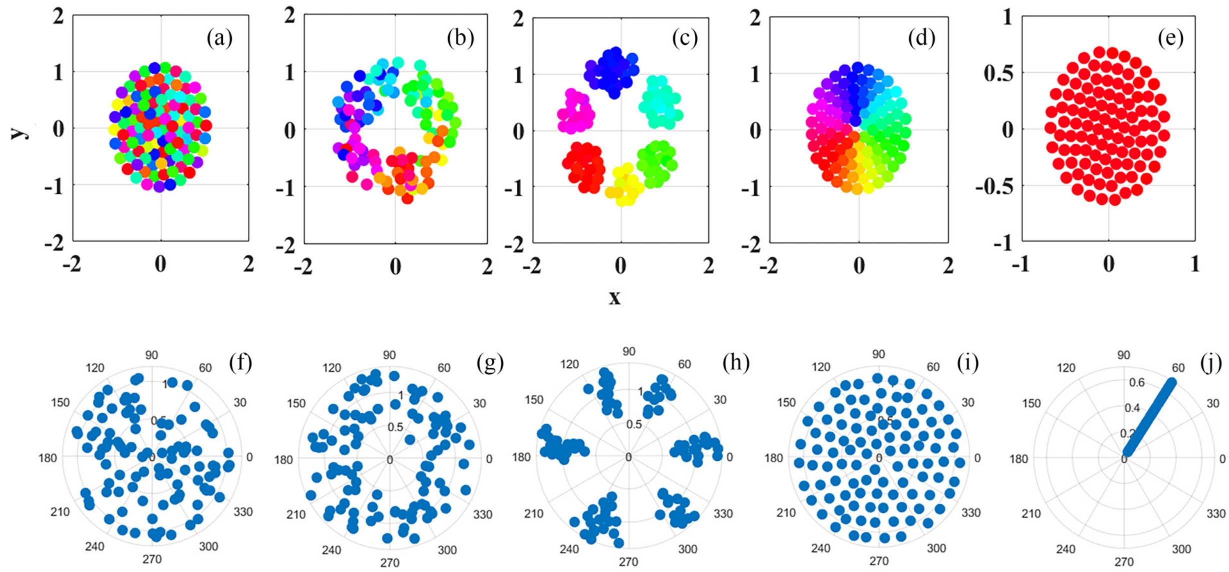

FIG. 1. Pattern formations of $N = 100$ swarmalators presented in Refs. [23,41] for different values of coupling parameters J and K. Scatter plot of the network and corresponding internal phase states, showing the following: (a) and (f) static async state ($J = 0.1,\ K = -1$); (b) and (g) active phase wave ($J = 1,\ K = -0.75$); (c) and (h) splintered phase wave ($J = 1,\ K = -0.1$); (d) and (i) static phase wave state ($J = 0.1, K = 0$); and (e) and (j) static sync state ($J = 0.1,\ K = 1$). Notice that the name of the phases refers to the spatial distribution of the agents.

This work highlights the effect of the internal dynamics' phase coupling strength on the spatial dynamics of swarmalators and vice versa. We investigated the influence of the natural frequencies of the swarmalators' internal dynamics during the transition to phase synchronization and found that explosive or first-order phase transition is not always a characteristic of these systems.

Consider a model of identical swarmalators confined to move in a 2D space like in O'Keefe *et al.* [23]. The position and the phase of each entity are coupled and described by Eqs. (1) and (2):

$$\dot{X}_i = v_i + \frac{1}{N}\sum_{j\neq i}^{N}[F_{\text{att}}(X_j - X_i)W(\theta_j - \theta_i) - F_{\text{rep}}(X_j - X_i)], \tag{1}$$

$$\dot{\theta}_i = w_i + \frac{K}{N}\sum_{j\neq i}^{N} H_{\text{att}}(\theta_j - \theta_i)G(X_j - X_i), \tag{2}$$

with $j = 1, 2, ..., N$; θ_i is the internal dynamics' phase of each swarmalator, represented by a Kuramoto-like model; $X_i = (x_i, y_i)^T$ is the position coordinates in the ith entity space, N is the population size of swarmalators; and v_i and w_i are the velocity and natural frequency of each element, respectively. We suppose that the system's dynamics is defined by the spatial angle ϕ, which describes its position defined by $\phi_i = \tan^{-1}(y_i/x_i)$, and the coupled phase θ_i.

The model presents an attractive and a repulsive behavior due to the existence of two interaction forces: a long and a short-range interaction, represented by the spatial interactions F_{att} and F_{rep}, respectively, and H_{att}, the phase interaction [23,26]. The competition between F_{att} and F_{rep} generates clusters of particles with sharp boundaries, like in Ref. [39]. The functions W and G represent the internal dynamics' influence on the oscillators' movement and vice versa, respectively. Thus, the model presented previously in Eqs. (1) and (2) can be rewritten as

$$\dot{X}_i = v_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{X_j - X_i}{|X_j - X_i|}(A + J\cos(\theta_j - \theta_i)) - B\frac{X_j - X_i}{|X_j - X_i|^2}\right], \tag{3}$$

$$\dot{\theta}_i = w_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j - \theta_i)}{|X_j - X_i|}. \tag{4}$$

In Eq. (3), the interaction between the oscillators in space is modulated by the term $A + J\cos(\theta_j - \theta_i)$ with $A = B = 1$. Let $v_i = v$ and $\omega_i = \omega$; therefore, $v = \omega = 0$. K represents the phase coupling, and J represents the attraction or repulsion between the system's particles. Positive values of the coupling parameter J lead to an attraction between particles with the same phase. Conversely, the opposite behavior is observed when $J < 0$ [23,26,40]. Summarizing, depending on the values of J and K, the swarmalators' dynamics vary from aggregation to synchronization, as shown in Fig. 1 [23–26,40,41].

The dynamics of swarmalators impose that they present only phase synchronization, since two agents cannot occupy the same spatial position but can nevertheless have the same internal phase. Per the current literature, we use the order parameter R defined [14,42–44] to characterize synchronization:

$$Re^{l\Phi} = \frac{1}{N}\sum_{i=1}^{N} e^{l\theta_i}, \text{ with } l^2 = -1. \tag{5}$$

When the internal dynamics of the ith and jth particle are synchronized, $\theta_i = \theta_j$ and the R norm tends to 1, otherwise

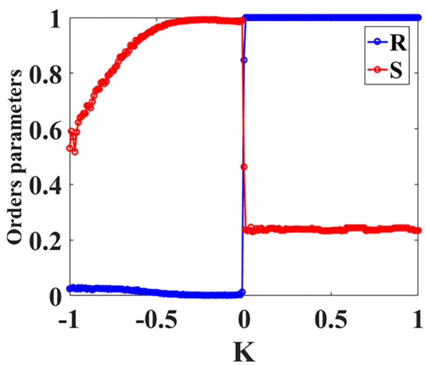

FIG. 2. Influence of internal dynamics' phase coupling K on the spatial dynamics of swarmalators represented by the order parameter, R, and the complex order parameter, S. The step of K is equal to 0,01 and spatial coupling $J = 1$ (See movie 1 in the Supplemental Material [47] where we show the evolution of the dynamics before and at the transition to synchronization as a function of K).

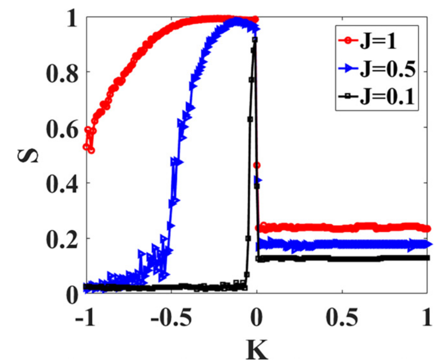

FIG. 3. Order parameter S versus coupling K, showing the evolution of the correlation between θ and ϕ for different values of the phase coupling J.

to 0. To measure the correlation between the spatial angle (angular position) ϕ and the internal phase dynamics θ, we define another order parameter S [23,24,26,40]:

$$S_{\pm}e^{l\Psi_{\pm}} = \frac{1}{N}\sum_{i=1}^{N} e^{l(\phi_i \pm \theta_i)}, \text{ with } l^2 = -1, \tag{6}$$

where $S = \max(S_+, S_-)$ is the real part of the complex order parameter. If $S = 1$ there is full correlation between ϕ and θ, and if $S = 0$ (or less than 1) it indicates a lack of correlation.

Consider a system of $N = 50$ units in a 2D space. The initial conditions of the positions are uniformly and randomly selected between $[-1, 1]$ with an initial phase θ between $[-\pi, \pi]$. To solve Eqs. (3) and (4) we use the fourth-order Runge-Kutta integration algorithm with an integration step of $dt = 0.05$. We study the long-term behavior reached by the swarmalator system after 20 000 iterations.

Figure 2 presents the Kuramoto order parameter's evolution as a function of the coupling parameter K showing an explosive transition from no synchronization between swarmalators for $K < 0$ to phase synchronization for $K > 0$.

The literature shows that the explosive transition affects the correlation between the spatial and internal dynamics of the swarmalators, which depends on the coupling constant K [23–26,45]. This correlation occurs in the splintered phase wave (SpPW) state because the spatial dynamics are highly influenced by the phases synchronizing in clusters before complete synchronization occurs. Figures 2 and 3 depict that, for $K > 0$, the correlation between swarmalators' spatial and phase dynamics is drastically reduced when the system synchronizes. For the SpPW and the active phase wave (APW) states, the swarmalators move around space [23,45,46] and we calculate that $S > 0$, the mean velocity V is nonzero and positive ($V > 0$) (with $V = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\dot{x}_i^2 + \dot{y}_i^2}$), and $R = 0$. $S = 1$, implying that the phase and the spatial angle are perfectly correlated, is only possible in the SpPW and SPW states shown in Figs. 1(c) and 1(h) and Figs. 1(d) and 1(i).

The range $K < 0$ is associated with the correlation range between phase and spatial dynamics ($S \approx 1$). Furthermore, the correlation S is also influenced by the spatial coupling strength J. Indeed for $K < 0$ and small values of J, there is no correlation between θ and ϕ (this means $S \simeq 0$) and the correlation appears when J increases (always for $K < 0$) (see Fig. 3). We observe no correlation for a synchronized state ($K > 0$).

To reach explosive synchronization for $J = 1$, the swarmalators move from the SpPW state [Figs. 4(a) and 4(c)] to the static wing phase wave (SWPW) state [Figs. 4(b) and 4(d)] [with properties similar to the static phase wave (SPW) state] before they get to the static sync (SS) state. The stability of this new state, an unexpected path to synchronization, is proven in the Supplemental Material [47–49], where we show that the state is stable in the sense of Sekieta and Kapitaniak [50] and Fermat and Solís-Perales [51]. The transition, from the SWPW state to the SS state, occurs at $K = 10^{-4}$ where the internal phase nodes are getting together in a cascade as time increases (Fig. 5); here, the number of independent states decreases with increasing time. The dynamics completing the transition can be seen in movie 3 of the Supplemental Material [47]. Note that the rotational motion of the phases illustrated in Fig. 4 persists until synchronization, despite the cascade effect shown in Fig. 5. This behavior is visible in movie 3 of the Supplemental Material [47].

To continue the study of the system's evolution towards the explosive transition to synchronization, we explore the energy of the swarmalator systems for the case of constant frequencies plotted in Fig. 6 as a function of K. We took advantage of previous works where we used the Hamiltonian formalism to understand the transition to synchronization in a star network of coupled oscillators [52], as well as to justify the existence of *chimera and multichimera states* [53].

Near synchronization, when the spatial distance between the swarmalators is practically constant and since the coupling in Eq. (4) is positive, the individual phase dynamics is that of a conventional *XY* model and the Hamiltonian becomes [46,54–58]

$$H_i = -\frac{K}{2N}\sum_{i\neq j}^{N}\cos(\theta_i - \theta_j). \tag{7}$$

The threshold value of the phase coupling K where the synchronization appears is $K_c = 0.005$ (and not zero as assumed before), and as shown in Fig. 6, the energy of the system

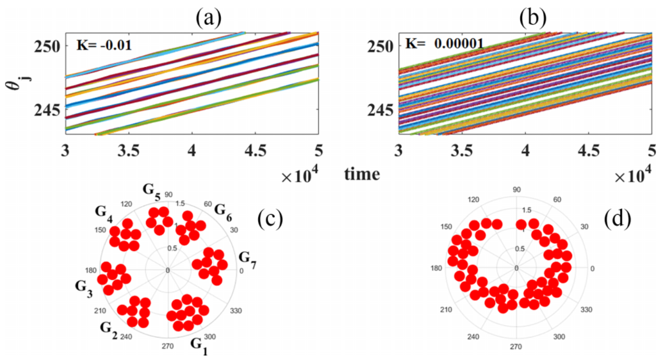

FIG. 4. Route to synchronization (the SpPW state, panels (a) and (c), is shown in movie 2 of the Supplemental Material [47]) where both the spatial evolution and the phase evolution are presented in a polar representation.

decreases when the swarmalators' phases synchronize. Since phase synchronization implies that $\theta_i = \theta_j$ (for all i and j), Eq. (7) becomes

$$H_c = -\frac{K_c}{2}. \quad (8)$$

From there, the minimum energy for which swarmalators synchronize is equal to H_c. This observation could explain that the apparently explosive transition to synchronization is in fact a process of energy loss where the elements synchronize to minimize the energy. Now let us look at the order of the phase transition as the system loses energy.

When the *XY* model represents the energy, Eq. (4) reduces to the well-known Kuramoto model [14]. Let us summarize how the present literature stands on this problem.

(a) According to Pazó [59], the Kuramoto model has a first-order transition from incoherence to synchronization in the thermodynamic limit, provided the natural frequencies are evenly spaced or uniformly distributed in a finite range.

(b) Gómez-Gardeñes *et al.* [27] showed that, when the natural frequency and a node's degree are equal, there is an explosive transition in a scale-free network of Kuramoto oscillators.

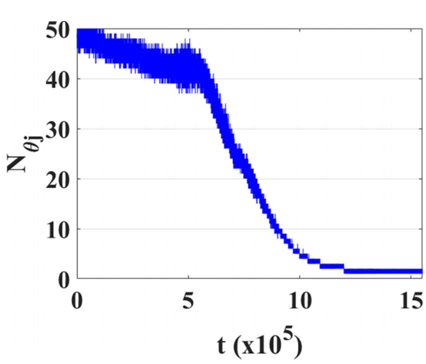

FIG. 5. Time dynamics of the swarmalators for $K = 10^{-4}$ showing the change of the number of nodes that are out of synchronization. Here the system moves from SWPW to SS states through the cascade route to synchronization at the critical value of K.

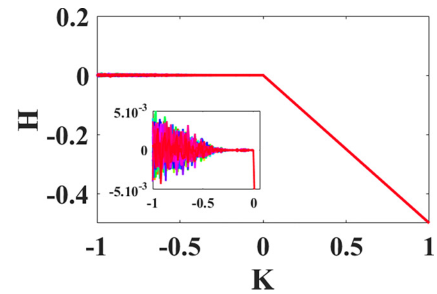

FIG. 6. Energy H_i of each entity of the system as a function of the phase coupling K.

(c) Skardal and Arenas [28] observed that adding infinitesimal disorder induces an explosive transition to synchronization for the case discussed in item (b).

(d) Leyva *et al.* [60] extended results and although they claim that "a sharp, discontinuous phase transition is not restricted to the above rather limited and apparently opposite cases [27,61], but it constitutes, instead, a generic feature of the synchronization of networked phase oscillators," their studies consider systems where the natural frequencies cannot be equal between themselves, or to the mean. This is neither our case, where we consider all natural frequencies equal to 0, nor that of Gómez Gardeñes *et al.*, which has a very low probability of not repeating values of frequencies.

(e) Hong and Martens [31] studied phase transitions in an *XY* model related to a variant of the Kuramoto model for coupled oscillators. They found that for the case without noise, the system shows features of a first-order phase transition at complete synchronization, while this transition is continuous for the noisy case.

As shown by Eqs. (3) and (4), our system belongs to the class discussed by Hong and Martens, and it reduces to an *XY* model with a first-order phase transition. Meanwhile, the problem of different natural frequencies is not so clear. To search for an explanation, we modified our system and studied three cases where natural frequencies are distributed as follows.

(a) The frequency is randomly distributed. Each oscillator is subject to a different frequency generated by a Rand function. From this random frequency distribution, we can see that, although the system synchronizes, the transition is not explosive (Fig. 7).

(b) The frequency is distributed in two groups of swarmalators. One group has a constant frequency while the other one is affected by noise, defined as follows ($w_1 = 0.005$):

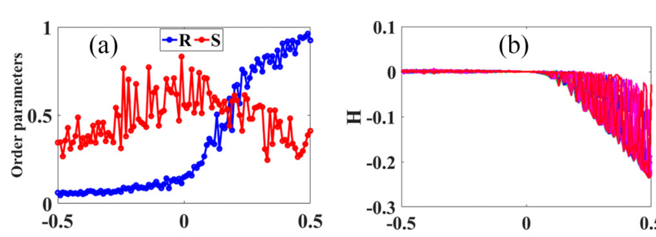

FIG. 7. Order parameter (a) and energy of swarmalators (b) as a function of phase coupling strength K for (case a).

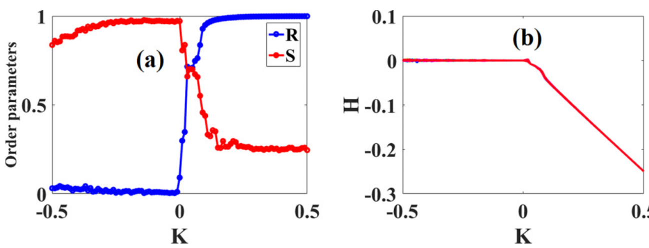

FIG. 8. Order parameter (a) and energy of swarmalators (b) as a function of phase coupling strength K for (case b).

(1) 1st group: from 1 to $\frac{N}{2}$, $w(i) = w_1$;

(2) 2nd group: from $\frac{N}{2}+1$ to N, $w(i) = w_1 + \varepsilon$, with $\varepsilon = 10^{-1}$.

Results are shown in Fig. 8, where we notice that the sudden transition has disappeared.

However, for this distribution of frequencies and for $K \approx -0.015$ the swarmalators are regrouped in clusters, meaning $R \approx 0$, $S \approx 1$, and $V \neq 0$, properties which normally describe the SpPW state [23]. In addition to these properties, these clusters are rotating (see snapshots in Fig. 9), where the internal phases of nodes are plotted in blue dots [panels (a1), (b1), and (c1)], while their spatial positions, in polar coordinates, are given in red dots [panels (a2), (b2), and (c2)] for three values of time.

Thus this dynamics is a rotational splintered phase wave (RSpPW) state characterized by the value 0 of the parameter $m_i = 0$ given by Eq. (6) in Ref. [62] and defined by its expression $m_i = 1 - 0.5(\max\{\cos[\phi_i(t)]\} - \min\{\cos[\phi_i(t)]\})$ for all nodes. Even if the total number of clusters in the considered case remains at seven, it appears that the number of nodes in the clusters can change with time.

(c) The frequency is distributed in three groups of swarmalators:

(i) 1st group: from 1 to 15, $w(i) = w_1$;

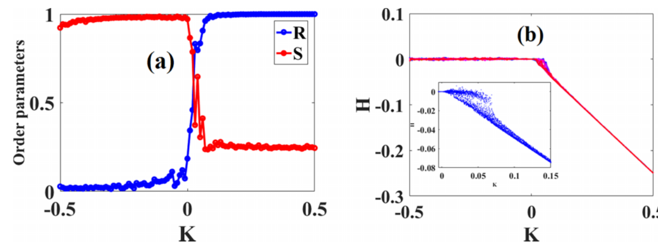

FIG. 10. Order parameter (a) and energy of swarmalators (b) as a function of phase coupling strength K for (case c).

(ii) 2nd group: from 16 to 31, $w(i) = w_1 + \varepsilon$, with $\varepsilon = 10^{-2}$;

(iii) 3rd group: from 32 to N, $w(i) = \text{rand}/10$.

We notice here that the phase transition towards synchronization is no longer explosive.

We studied the transition to synchronization in this case in detail and found that it is not a first-order phase transition. First the oscillators cluster and then they merge into asymptotic synchronization, as can be seen in Fig. 10(b) with emphasis on the inset.

Therefore, while presenting here some possible synchronization cases in swarmalators, a thorough study of this and related problems on phase transitions in systems of mobile oscillators and their representation by the Kuramoto model is necessary.

Summarizing, in this work, we have studied the behavior of systems where phase and spatial dynamics are coupled, called swarmalators. The effect of positive and negative phase coupling strength on the system's dynamics under the impact of initial conditions was shown. As proposed in Ref. [23], the five steady states of swarmalators were characterized, and it was found that the system maps into an XY model with an explosive transition to the synchronization when the system is subject to an attractive ($K > 0$) and repulsive ($K < 0$) phase coupling. Based on the conventional mean-field XY model, the

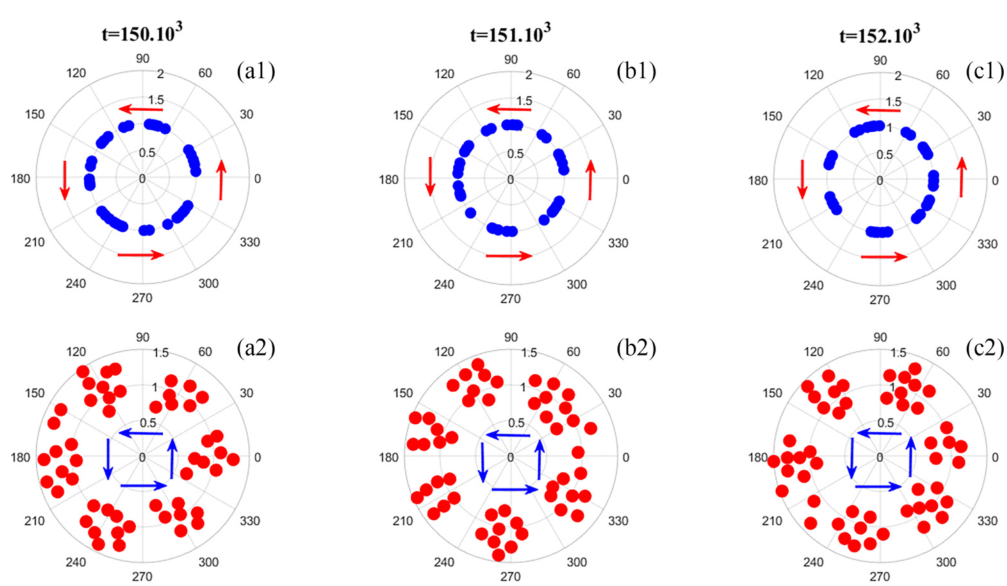

FIG. 9. Snapshots of all internal phases [panels (a1), (b1), and (c1)] and nodes positions [panels (a2), (b2), and (c2)] showing the changes in the positions of clusters for $t = 150 \times 10^3$, $t = 151 \times 10^3$, and $t = 152 \times 10^3$ for (case b).

mean energy of a system using the Hamiltonian formalism has been evaluated. This evaluation highlights that the transition of swarmalators to synchronization can be explained as a process of energy loss which leads them to synchronize when the critical value $H_c = -\frac{K_c}{2}$ reaches $K_c = 0.005$, to minimize this energy. During this study, some expected results were obtained like the second-order transition and pattern formation as described in Ref. [23]. However, in addition to the new state as the rotational splintered phase wave, another unexpected result here is the existence of first-order transition without correlation between the natural frequencies of elements and the degree of their nodes. Indeed, this kind of transition, which was shown previously to happen when the natural frequencies of the oscillators are equal to the number of links they possess [27], always occurs when the natural frequencies of all oscillators are equal, but it disappears when they show complete or partial disorder. To better understand why the first-order transition occurs, we studied three main cases of natural frequency distribution. This hypothesis showed that the way natural frequencies are distributed can significantly affect the type of transition in swarmalator systems. By studying the pretransition dynamics in detail, we found that the internal phases undergo a rotational wave state, the appearance of which we were unable to explain. This will be the subject of further work. By making a detailed investigation of the dynamics before the transition, we found that the internal phases undergo a rotational wave state, which we have not been able to explain why it appears there. More on this point will be done in future work.

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